Physics I
ISI B.Math
Midterm Exam : March 5, 2013

Total Marks: 60

Answer question 1 and any 4 from the rest

1.(**Marks** $= \mathbf{2} \times \mathbf{6} = \mathbf{12}$) Choose the correct option.

(i) A particle P moves so that its position vector $\mathbf{r}$ satisfies the differential equation $\dot{\mathbf{r}} = \mathbf{c} \times \mathbf{r}$ where $\mathbf{c}$ is a constant vector . Which of the statements about the motion of the particle is false ?

(a) Kinetic energy is not conserved.
(b) Angular momentum about the origin is conserved.
(c) The particle is moving in a circular orbit.

(ii) S is an inertial frame. S' is another inertial frame moving with a velocity $\mathbf{v}$ with respect to S. For which of the following quantities will an observer in S and an observer in S' disagree on their measurements ?

(a) The force $\mathbf{F}$ acting on a particle of mass m
(b)The mutual potential energy $U(\mathbf{r})$ of two particles interacting through a gravitational force , where $\mathbf{r}$ is the relative position vector of the two particles.
(c) The total energy (kinetic + potential) of the two particles mentioned in option (b).

(iii) A particle is moving in three dimensions under the influence of a central force which gives rise to a potential $U(r) = \frac{1}{2}kr^2$ where k is a constant. Which of the following statements about the motion of the particle is false ?

(a) A possible trajectory for the particle is a circular orbit.
(b) The particle can have bounded or unbounded motion depending on its total energy.
(c)The radius vector of the particle sweeps out equal areas in equal times.
(d)The total energy of the particle is conserved.

(iv) A block of mass resting on a table m is attached to a spring of spring constant k and $\omega_0 = \sqrt{\frac{k}{m}}$. The other end of the spring is fixed to the wall. The block is subject to a frictional force $= -2m\beta\dot{x}$ ($\beta < \omega_0$) and is released from rest from a position $x = A$. Which of the following is a possible correct solution for the subsequent motion of the block?

(a) $x(t) = Ae^{-\beta t}e^{-\sqrt{\omega_0{}^2-\beta^2}t}$
(b) $x(t) = Ae^{-\beta t}e^{\sqrt{\omega_0{}^2-\beta^2}t}$
(c) $x(t) = Ae^{-\beta t}\cos(\sqrt{\omega_0{}^2-\beta^2}t)$
(d) $x(t) = Ae^{-\beta t}\sin(\sqrt{\omega_0{}^2-\beta^2}t)$

(v) Which of the following is a conservative force?

(a) $A\dot{x}\,\hat{\mathbf{x}} + By\,\hat{\mathbf{y}}$ where A and B are constants.
(b) $A(t)r^2\,\hat{\mathbf{r}}$ where $A(t)$ is an arbitrary function of t.
(c) $Ay\,\hat{\mathbf{x}} + Bx\,\hat{\mathbf{y}}$ where A and B are constants.
(d) $f(r)\,\hat{\mathbf{r}}$, where $f(r)$ is an arbitrary function of r.

(vi) A ball bearing of mass m is released from rest in a vertical column of castor oil which exerts a retarding force $= -mkv$ on the ball bearing. Which of the following expressions can possibly correctly describe its velocity at time t?

(a) $v = \frac{g}{k}(1 - e^{-kt})$
(b) $v = \frac{g}{k}(1 - e^{kt})$
(c) $v = \frac{g}{k}e^{-kt}$
(d) $v = \frac{mg}{k}(1 - e^{-kt})$

2. (**Marks = 4 + 8 = 12**)

(a) A particle with polar coordinates r, θ which are functions of time t is moving in a plane. The velocity and the acceleration of the particle can be written as $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta}$ and $\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\theta}$. Find $v_r, v_\theta, a_r, a_\theta$ as a function of $r, \theta, \dot{r}, \dot{\theta}$.
(b) A bee flies on a trajectory such that its polar coordinates at time t are given by $r = \frac{bt}{\tau^2}(2\tau - t), \theta = \frac{t}{\tau}, (0 \le t \le 2\tau)$, where b and τ are positive constants. Find the velocity vector of the bee at time t. Show that the least speed achieved by the bee is $\frac{b}{\tau}$. Find the acceleration of the bee at this instant.

3. (**Marks = 8 + 4 = 12**)

(a) An overdamped harmonic oscillator satisfies the equation

$$\ddot{x} + 10\dot{x} + 16x = 0$$

At time $t = 0$, the particle is projected from the point $x = 1$ toward the origin with speed u. Find $x(t)$. Show that the particle will reach the origin at some later time t if

$$\frac{u - 2}{u - 8} = e^{6t}$$

How large must u be so that the particle will pass through the origin?

(b) An undamped driven harmonic oscillator satisfies the equation of motion $m\ddot{x} + {\omega_0}^2 x = F(t)$ where the driving force $F(t) = F_0 \sin(\omega t)$. The oscillator is released at rest from the position $x = A$. Find a *particular*(not general) solution for this inhomogeneous equation. How will this solution change if the oscillator were released instead from the position $x = 0$ with a velocity $v = v_0$?

4. (**Marks = 8 + 4 = 12**)

(a) Consider a particle of mass m whose motion starts from rest in a constant gravitational field. If a resisting force proportional to the square of the velocity (i.e, kmv^2) is encountered, show that the distance s the particle falls in accelerating from v_0 to v_1 is given by

$$s(v_0 \to v_1) = \frac{1}{2k} \ln \left[\frac{g - kv_0^2}{g - kv_1^2} \right]$$

(b) A particle of mass m and charge q is moving under the influence of a constant magnetic field $\mathbf{B} = B_0\mathbf{k}$, where B_0 is a constant. At $t = 0$, the velocity vector of the particle lies in the $x - y$ plane. Show that the subsequent motion of the particle will be confined to the $x - y$ plane. The particle starts out in the position $(r = a, \theta = 0)$ and reaches a position $(r = a, \theta = \frac{\pi}{2})$ at a later time $t = T$. Find the change in the kinetic energy of the particle between these two positions.

5. (**Marks = 7 + 5 = 12**)

(a) A particle moves under the influence of the potential energy $V(x) = A/x^2 - B/x$ where $A, B > 0$ and $x \geq 0$. Make a rough sketch of the potential energy. Find the range of energies for which there will be bounded and unbounded motion and find the frequency of small oscillations about the equilibrium point. If $A, B < 0$, will it be possible to find the frequency of small oscillations about the equilibrium point?
(b) A particle moves in the following trajectory : $x = A\cos\omega_1 t : y = A\sin\omega_2 t$ where ω_1 and ω_2 are constants. Find the relationship between ω_1 and ω_2, such that the force acting on the particle will be a central force.

6. (**Marks = 4 + 4 + 4 = 12**)

(a) Find the force law for a central force field that allows a particle with angular momentum l to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k andα are constants.
(b) Show that the total energy of the orbit is zero , given that the potential energy $U(r = \infty) = 0$.
(c) Determine $r(t)$ and $\theta(t)$ for the particle. Your answer will involve an arbitrary integration constant to be determined from initial conditions.

Information you may or may not need:

$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$

For a particle of mass m and angular momentum L moving under the influence of a central force $\mathbf{f}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$ and $u = \frac{1}{r}$ the path equation is given by

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f(\frac{1}{u})$$